REVISTA MEXICANA DE ECONOMÍA Y FINANZAS
Nueva Época
REMEF
(THE MEXICAN JOURNAL OF ECONOMICS AND FINANCE)

Revista Mexicana de Economía y Finanzas, Nueva Época

Volumen 21 Número 1, enero - marzo 2026, pp. 1-17, e1494

DOI: https://doi.org/10.21919/remef.v21i1.1494

(Recibido: 27/diciembre/2024, aceptado: 25/abril/2025, publicado: 1/enero/ 2026)

imef fundación de investigación

Relativistic Black–Scholes Equation

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Abstract

The Relativistic Black Scholes Model presented in this paper is a generalization that is not very well known since the original version of 1973, because its effects are still not very significant. Actually, any small advantage in information knowledge in the High-Frequency Trading can become great arbitrage opportunities. In order to determine the value of a financial option, it is necessary to construct a density function associated with the determination of prices of financial assets using the concepts of quantum mechanics and relativity gathered in the Dirac equation. For the effect to be appreciated, the distance between traders would have to be on a large scale, but an equivalent concept can be found, which is the speed of light of the market, which rather involves delays in the technology used and human reaction times. At the end, an approximation is made of this implicit velocity parameter.

JEL Classification: G130, G140, C6
Keywords: Financial Options, Black Scholes, relativity, trading, high-frequency

Ecuación Black-Scholes relativista

Resumen

El modelo de Black Scholes presentado en este trabajo es una generalización a la versión relativista que no es muy conocida desde a la versión original de 1973, debido a que sus efectos aún son poco significativos. En el trading de alta frecuencia (High-Frequency Trading) de estos tiempos cualquier pequeña ventaja en el conocimiento de la información puede convertirse en grandes oportunidades de arbitraje. Para poder determinar el valor de una opción financiera es necesario construir una función de densidad asociada a la determinación de precios de activos financieros utilizando los conceptos de la mecánica cuántica y la relatividad reunidos en la ecuación de Dirac. Para que el efecto pueda ser apreciado, la distancia entre operadores tendría que ser a gran escala, pero puede encontrarse un concepto equivalente que es la velocidad de la luz del mercado que involucra más bien los retrasos en la tecnología utilizada y los tiempos humanos de reacción. Al final se hace una aproximación de este parámetro implícito de la velocidad.

Clasificación JEL: G130, G140, C6
Palabras clave: Opciones, Black Scholes, relatividad, trading, alta frecuencia

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*No funding source was received for the development of this research.

1. Introduction

In the actual world in various financial systems, whether stock markets, currency markets, derivatives markets, etc., the speed of information communication has become an important factor in obtaining arbitrage opportunities in the face of market inefficiencies. Technological developments in communication and increasingly efficient algorithms make it necessary to develop more comprehensive and general theories.

The velocity of operations can produce unexpected events , as was the case of the Flash Crash on May 6, 2016, where the Dow Jones Industrial Average (DJI) had its biggest drop in a day (nearly 9%) but also made a surprising recovery in the same day. To avoid sudden crises like the one above, intensive research is needed into how markets work and the complex interaction of algorithms and humans in order to try to counteract possible disasters.

In the field of financial derivatives, the Black-Scholes equation has been the obligatory reference for the valuation of financial options, although its limitations are known, for example, in terms of volatility considerations and its distributions associated with prices, and therefore there have been attempts at generalizations in various theoretical directions.

The general aim of this work is to generalize the Black-Scholes equation, taking into account a necessary correction in the information speed factor. According to the laws of physics, despite the speed of light being very high (300,000 km/s), it is a limitation in the communication of information, in other words, it cannot occur instantaneously, and this speed, under certain conditions, could represent a small advantage in intraday financial trading.

This paper begins with a review of the main articles on brownian motion, financial options and special relativity. The second part shows some of the important developments taking place in the world of high-frequency trading and the importance of information speed (factors influencing prices). In the model section, we made a review of the theoretical framework of the Black-Scholes equation, as well as the concepts and ideas of quantum mechanics and relativistic physics to arrive at the relativistic version. In the section on the proposed model, the deduction of the relativistic Black-Scholes equation is proposed. At the end, the results of the equivalent of the relativistic model are presented, as well as the parameter of the speed of light implicit in the market, and the conclusions.

2. Literature Review

This section shows some of the main works related to the subject of brownian motion and Black-Scholes option valuation, high-frequency trading linked to the subject of special relativity. In Kurianovich E. A. et al (2024) an approximation is made to the theory of a relativistic random process that considers its use through the method of path integral considering brownian motion and taking into account a limit on speed. The authors constructed a relativistic analog of Wiener's measure as a weak limit of finite difference approximations and proposed a formula for calculating the transition probability of particles during relativistic brownian motion. The calculations were performed using different methodologies and similar results were obtained.

In Blaho R. et al (2022), the authors mentioned that the Black-Scholes expression, published in 1973, provided financial professionals with a mathematical method for determining prices for European options, which was used successfully until the so-called Black Monday of 1987, when crucial discrepancies with real prices occurred. The authors propose that relativistic physics is capable of helping to solve these problems, obviously at a cost. In their work, they try to emphasize the role of relativistic physics in the field of financial mathematics and to propose some observations and ideas that could help this field in its further development.

A classic work on the subject is that of Dunkel J.et al (2009), according to the author, the mathematical description of stochastic processes has led to new approaches in other fields. Within the framework of special relativity, the authors review recent advances in the phenomenological description of relativistic diffusion processes. According to the author, given that the speeds of relativistic particles are limited by the speed of light, non-trivial relativistic Markov processes in space-time do not exist; in other words, the relativistic generalizations of the non-relativistic diffusion equation and its Gaussian solutions must necessarily be non-Markovian.

In Trzetrzelewski M. (2017) the Black-Scholes equation is revised and after a coordinate transformation it finds its equivalence to the heat equation and also its relativistic extension with the known telegrapher equation, which can also be derived from the Euclidean version of the Dirac equation. Therefore, the relativistic extension of the Black-Scholes model, particularly in the case of European options, derives quite naturally from relativistic quantum mechanics.

Like the ideas of the triumph of relativistic mechanics over classical mechanics when speeds approach the speed of light, the authors explore an improvement in Black-Scholes option prices and a respective solution in Qu Y. et al (2017).

The author shows a solution with a significant improvement over other solutions and obtains a new closed-form option pricing formula, which contains the speed limit of information transfer (the speed of light c) as a new parameter. It also shows how the new formula converges with the Black-Scholes formula when c tends to infinity, the new formula can flatten the well-known volatility smile, which is more consistent with empirical observations.

In Carvalho V.H. et al (2021) is mentioned that the exchange of information is increasingly faster and will eventually approach the speed of light. Due to advances in high-frequency trading, the author suggests the need to consider the effects of the theory of relativity on financial models. Time and space, in certain circumstances, are not dissociated and cannot longer be interpreted as Euclidean. The paper provides an overview of research in this field while formally defining the key notions of space-time, proper time and an understanding of how time dilation affects financial models. They also illustrate how special relativity modifies the price of options and hedging, according to the Black-Scholes model, when market participants are in two different frames of reference. In particular, the relativistic effects of maturity and volatility are analyzed.

According to Romero J.M. et al (2016), the Klein-Gordon equation is used to propose a generalized Black-Scholes equation. In their work, they find that in the limit the generalized equation is invariant under conformal transformations, in particular invariant under scale transformations. In this limit, it is shown that the distribution of stock prices is given by a Cauchy distribution, rather than a normal distribution.

In Johnson J. et al (2012) makes it clear that society's drive towards ever faster, more technical systems makes it urgent to understand more and more the threat posed by “black swan” extreme events .

According to the author, in May 2010, it took just five minutes for a spontaneous mix of human and machine interactions in global commercial cyberspace to generate an unprecedented system-wide Flash Crash. However, little is known about what lies ahead in the event that humans become unable to respond or intervene quickly enough. Their work analyzes thousands of ultra-fast black swan events that have been discovered in stock price movements between 2006 and 2011.

Wissner Gross A.D. (2010) mentions that recent advances in high-frequency financial trading have made slight delays in propagation between geographically separated exchanges relevant. The authors show that there are optimal locations from which to coordinate statistical arbitrage of spatially separated security pairs and calculate an ideal map representative of such locations on Earth. Furthermore, local trading along chains of intermediate locations results in a novel effect, in which the relativistic propagation of tradable information is effectively retained or stopped by arbitrage.

3. Stylized facts

According to Buchanan (2015), technological advantages are becoming increasingly important in competition in trading or financial trading, both in the number of operations (100,000 per second) and in the speed of communication, for example with the use of fiber optics where information travels at 2/3 of the speed of light in a vacuum. It is worth mentioning the time taken for operations between London and New York, which is 2.6 milliseconds, or a network of military-grade lasers that has been installed to connect the financial centers of New York, New Jersey, London and Frankfurt. In other words, it is important to consider the advantages that technology offers.

High-frequency trading relies on the speed of algorithms and computers to know when to buy and/or sell and feeds on the same market data. However, some companies claim that there is unequal access to extreme speed that erodes trade equity among participants. Besides High-frequency trading provides liquidity to markets, making it easier for investors to find trading partners at reasonable prices and also to equalize the difference between the prices at which one can buy or sell from the markets, that is, it helps to synchronize prices in all markets.

According to Buchanan (2015), the nature of financial market operations is very different today to how they were in the past, rather than reflecting the collective decisions of people, they combine the behavior of complex networks, electronic communications and their interactions with humans.

The problem is potentially increasing globally as high-frequency trading has moved to international markets for derivatives and other assets and practically including all industries, including energy and food, insurance and banking.

Therefore, in global financial markets there is an urgent need to develop important strategic models (the derivatives market) and predictive simulation capabilities, comparable to global-scale meteorological monitoring.

4. The Model

In the next section we will start by recalling the basic concepts of financial options and the Black-Scholes equation. In the second part we will review the equivalence between this equation and another fundamental physics equation, the Schrödinger equation. Subsequently, the Dirac equation, or relativistic version of the latter equation, will be derived, but first there will be a brief introduction to the concepts of special relativity.Option contracts are generically known as derivatives or contingent liabilities. Remember that an option is a contract that gives the holder the right, but not the obligation, to buy or sell a certain asset at an agreed price at a fixed point in the future. In particular, an european call option with an exercise price K, expiring at T and on an underlying S, is a contract that gives the right to buy the underlying asset at a price K at time T; in the case of a put option, it gives the right to sell.

The Black-Scholes equation (Black, Scholes (1973)) is a parabolic partial differential linear equation with values on the boundary. This equation assumes that the behavior of the underlying associated with the derivative can be modeled in its stochastic term with a Brownian motion and for its derivation it makes use of at least two basic finance concepts: hedging and no-arbitrage.

The reduction of uncertainty or randomness is known as hedging and an elimination of risk using a portfolio of two instruments (an option and its underlying) is known as delta hedging. Once the risk-free return of the previous portfolio is known, it should be equal to the return of a bank account paying a certain fixed interest rate. For the deduction of the Black-Scholes model, we arrive at[2]:

$$\frac{\partial C(S,t)}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C(S,t)}{\partial S^2} + rS\frac{\partial^2 C(S,t)}{\partial S} - rC(S,t) = 0 \qquad (1)$$

where C= C(t, S) is the derivative of an underlying S with boundary condition, for call and put :

$$C(t,S) = Max(S-K,0) \qquad (2\text{ a})$$
$$P(t,S) = Max(K-S,0) \qquad (2\text{ b})$$

whose solution for european call and put option at any $0 \leq t \leq T$ strike price K, maturity T, variance σ and risk-free rate r, is given by:

$$C(S(t),t) = S(t)N(d_1) - Ke^{-r(T-t)}N(d_2) \qquad (3\text{ a})$$
$$P(S(t),t) = Ke^{-r(T-t)}N(-d_2) - S(t)N(-d_1) \qquad (3\text{ b})$$

where

$$d_1 = \frac{\ln\left(\frac{S(t)}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}} \qquad (4\text{ a})$$
$$d_2 = \frac{\ln\left(\frac{S(t)}{K}\right) + \left(r - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}} \qquad (4\text{ b})$$

[2] With the assumptions: risk-free rate and constant volatility, no transaction costs and no arbitrage opportunities

On the other hand, one of the fundamental equations of physics in the area of quantum mechanics that describes the behavior of a free particle is the Schrödinger equation expressed below (following the work of Romero J.M. et al (2016)).

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} \qquad (5)$$

where m is the mass of the particle, $\hbar$ is Planck's constant and $\psi(x,t)$ is the wave function (which is the basis for determining the probability of position of a subatomic particle). If we apply the following mapping, $\tilde{t} = it, \hbar = 1,\ m = \frac{1}{\sigma^2},\ x = lnS$ then we have

$$\psi(x,t) = e^{-\left(\frac{1}{\sigma^2}\left(\frac{\sigma^2}{2}-r\right)x+\frac{1}{2\sigma^2}\left(\frac{\sigma^2}{2}+r\right)^2 t\right)}C(x,t) \qquad (6)$$

In other words, we can go from Schrödinger's free particle equation to the Black Scholes equation.

Considering that the case $m \longrightarrow 0$ it is not studied within traditional quantum mechanics, but it is within relativistic quantum mechanics, the Schrödinger equation is transformed into the well-known Klein-Gordon equation

$$-\frac{\hbar^2}{c^2}\frac{\partial^2\psi(x,t)}{\partial\tilde{t}^2} + \frac{\partial^2\psi(x,t)}{\partial x^2} - m^2c^2\psi(x,t) = 0 \qquad (7)$$

where c is the speed of light, and we note that when $m \longrightarrow 0$ the Klein-Gordon equation makes sense and when the Schrödinger equation is recovered. On the other hand, following the following mapping , $\tilde{t} = it, \hbar = 1, m = \frac{1}{\sigma^2}, x = lnS, c^2 = q$

$$\psi(x,t) = e^{-\left(\frac{1}{\sigma^2}\left(\frac{\sigma^2}{2}-r\right)x+\left(\frac{1}{2\sigma^2}\left(\frac{\sigma^2}{2}+r\right)^2-\frac{q}{\sigma^2}\right)t\right)}C(x,t) \qquad (8)$$

we arrive at the also known Klein - Gordon equation

$$\frac{1}{q}\frac{\partial^2 C(S,t)}{\partial t^2} \quad \left(\frac{2}{\sigma^2} - \frac{1}{q\sigma^2}\left(\frac{\sigma^2}{2}+r\right)^2\right)\frac{\partial C(S,t)}{\partial t} + S^2\frac{\partial^2 C(S,t)}{\partial S^2} + \frac{2r}{\sigma^2}S\frac{\partial C(S,t)}{\partial S} +$$
$$+\left[\frac{1}{4q\sigma^4}\left(\frac{\sigma^2}{2}+r\right)^4 - \frac{2r}{\sigma^2}\right]C(S,t) = 0 \qquad (9)$$

which can be rewritten as follows

$$\frac{\sigma^2}{2q}\frac{\partial^2 C(S,t)}{\partial t^2} + \left(1-\frac{1}{2q}\left(\frac{\sigma^2}{2}+r\right)^2\right)\frac{\partial C(S,t)}{\partial t} = -\frac{\sigma^2}{2}S^2\frac{\partial^2 C(S,t)}{\partial S^2} - rS\frac{\partial C(S,t)}{\partial S} + rC(S,t) \qquad (10)$$

equation (10) in the limit when $q \longrightarrow \infty$ we recover the Black Scholes equation. In the following paragraphs, some of the basic concepts of special relativity will be recalled. In order to be able to propose a relativistic quantum mechanics equation, the deduction will be followed according to Trzetrzelewski, M. (2017)).

The special theory of relativity is a result of the facts of natural things, which are not observable in classical mechanics and is based on two postulates[3]

[3] 3 The special theory of relativity was developed by Albert Einstein and revolutionized the traditional concepts of classical physics Einstein A.(1905)

i) The principle of relativity, which states that the laws of nature are invariant in all inertial reference frames
ii) The principle of constant velocity and specifically that the speed of light in a vacuum is finite and equal in all reference frames

The invariance of natural laws in all inertial reference frames comes from the observation that in all such frames, all experiments with these laws have the same results[4] .

It is important to determine the relationship between a reference system S and one S´ that moves at a constant velocity ν relative to the first. Let us suppose that the movement, without losing generality, is in the particular direction of the X axis, which has traditionally been explained by Galileo's transformations. More generally, as a consequence of the postulates in special relativity, it is given by the Lorentz transformation, and corresponds to the following expressions for the relationship between position and time (Blaho R. (2021)):

$$x' = \gamma(x - vt) \tag{11}$$

$$t' = \gamma\left(t - \frac{vx}{c^2}\right) \tag{12}$$

with $\gamma = \frac{1}{\sqrt{1-\frac{\nu^2}{c^2}}}$

In tensorial terms, a vector in this space known as Minkowski space can be defined by a contravariant vector

$$x^\alpha = (x^0, x^1, x^2, x^3) = (ct, x, y, z)$$

It is the contravariant version of a vector called an event, the covariant version is defined by the following:

$$x_\alpha = g_{\alpha\beta}x^\beta$$

where $g_{\alpha\beta}$ represents the metric tensor in Minkowski space. On the other hand, if we assume that we have a clock at the origin of the reference system S and in S´ and at the beginning they coincide at t = t´ = 0 , given a certain signal the system S´ begins to move with ν velocity in the direction X, for to simplify and without losing generality. Now a time interval from the beginning of translation between S and S´

$$\Delta t = \gamma \Delta t' \tag{13}$$

and proper time $\Delta\tau = \Delta t'$ is defined as the invariant quantity that always represents the shortest time compared to all possible reference times. The special meaning in special relativity is the so-called space-time interval, which represents the distance between two events in Minkowski space, which is an invariant in Lorenz space.

$$(\Delta S)^2 = (c\Delta t)^2 - \Delta^2 x - \Delta^2 y - \Delta^2 z \tag{14}$$

$$(\Delta S)^2 = (c\Delta\tau)^2 \tag{15}$$

With a constant speed of light, it gives another useful formulation of prototype time that is also invariant $\Delta\tau = \Delta\tau'$ in spacetime for any inertial reference system. In Minkowski space, the quadri-acceleration of a particle is defined where ds is the infinitesimal version of the original spacetime ΔS

$$u^\alpha := \frac{dx^\alpha}{d\tau} = c\,\frac{dx^\alpha}{ds} \tag{16}$$

and the second derivative with respect to proper time gives us the quadri-acceleration

[4] The second postulate was verified with the results of the Michelson-Morley experiment of 1887.

$$a^{\alpha} := \frac{du^{\alpha}}{d\tau} = c\,\frac{du^{\alpha}}{ds} \quad (17)$$

and also for the quadri-momentum

$$p^{\alpha} := m_0 u^{\alpha} = m_0(\gamma c, \gamma v) \quad (18)$$

using the expression for the mass of a particle

$$E = mc^2 = m_0 \gamma c^2 \quad (19)$$

and we obtain another formulation used

$$p^{\alpha} = \left(\frac{E}{c}, \boldsymbol{p}\right) \quad (20)$$

and we have the moment of the quadri-moment

$$p^{\alpha} p_{\alpha} = \frac{E^2}{c^2} - p^2 = m_0^2 u^{\alpha} u_{\alpha} = m_0^2 c^2 \quad (21)$$

where we use the fact that the norm of the quadri-velocity is equal to c^2 and we obtain the famous energy-momentum relation

$$E^2 = p^2 c^2 + m_0^2 c^4 \quad (22)$$

In the following paragraphs, based on the previous sections, the Dirac equation is deduced (following the version of Estevez-Fernandez J., (2007)) which is the equivalent of the relativistic Schrödinger equation. We know that the Schrödinger equation is also a diffusion equation with a purely imaginary diffusion coefficient, if we rewrite the equation in this way

$$i\hbar \partial_t \psi = H\psi \quad (23)$$

where the energy operator is defined as

$$\hat{E} = i\hbar \partial_t \quad (24)$$

and the Hamiltonian considering the kinetic energy term is

$$H = \frac{p^2}{2m} = -\frac{\hbar^2}{2m} \nabla^2 \quad (25)$$

the Schrödinger equation is not relativistic and considers the first-order time derivative and the second-order spatial derivative, while in Special Relativity, as the coordinates play the same role, the derivatives must be of the same order, so the following options are available: (Estevez-Fernandez J., (2007))

1. Klein-Gordon equation: Both derivatives are second order

$$(\partial_{\mu}^2 + \mu^2)\psi \equiv \left(\boxed{\partial_{\mu}^2}^{\,2} + \mu^2\right)\psi \quad (26)$$

with the problem that probability $|\psi|^2$ can become negative and energy E can also become negative.

2. Dirac equation: where both derivatives are first order

$$(\partial_{\mu} + \mu^2)\psi = 0 \quad (27)$$

with the drawback that energy can be negative (and as a consequence antimatter can appear). We will now review the second case, which corresponds to Dirac's equation. The energy-momentum relation in Special Relativity gives the invariant of the quadri-momentum (which we already had in equation (22)), substituting the quantum operators knowing that the energy operator is defined by the Schrödinger equation gives

$$(i\hbar \partial_t)^2 = (-i\hbar c \nabla)^2 + (mc^2 \mathbb{I})^2 \quad (28)$$

the Schrödinger equation tells us that the second member is the Dirac term, Hamiltonian squared, where the Dirac equation is simply the Schrödinger equation whose Hamiltonian is the Dirac Hamiltonian, that is,

$H^2 = (-i\hbar c\nabla)^2 + (mc^2\mathbb{I})^2$

where the most general Hamiltonian that fulfills the previous equation is:

$$H = \bar{\alpha}\,(-i\hbar c\nabla) + \beta mc^2\mathbb{I} \quad (29)$$

where $\bar{\alpha} \equiv (\alpha_{1,}\alpha_{2,}\alpha_3)$ is not necessarily a vector but has 3 components β and is not necessarily a scalar but has 1 component (under Lorentz transformations). Then

$$H^2 = \bar{\alpha}\,\bar{\alpha}\,(-i\hbar c\nabla)\cdot(-i\hbar c\nabla) + mc^2(\bar{\alpha}\,\beta + \beta\bar{\alpha}\,)(-i\hbar c\nabla) + \beta^2(mc^2)^2 \quad (30)$$

performing the scalar products and separating, we arrive at

$$H^2 = \sum_{i=1}^{3}\alpha_i^2\left(-i\hbar c\partial_{x_i}\right)^2 + \sum_{j<i=1}^{3}\left(\alpha_i\alpha_j + \alpha_i\alpha_j\right)\left(-i\hbar c\partial_{x_i}\right)\left(-i\hbar c\partial_{x_i}\right) + \sum_{i=1}^{3} mc^2(\alpha_i\beta + \beta\alpha_i)\left(-i\hbar c\partial_{x_i}\right) + \beta^2(mc^2)^2 \quad (31)$$

comparing the equations we obtain the conditions that the elements we have introduced in the Dirac Hamiltonian have to fulfill. The elements that fulfill these properties belong to a Clifford algebra:

1.$\alpha_i^2 = \beta^2 = 1$ (32 a)

2.$[\alpha_i, \alpha_j] = 0. i \neq j$ (32 b)

3$[\alpha_i, \beta] = 0$ (32 c)

being a type of matrix that we will call Dirac gamma matrices, defining the Dirac gamma matrices we have

$\gamma^0 \equiv \beta, \gamma^i \equiv \beta\alpha_i$

where $\gamma^0 = \begin{pmatrix}\mathbb{I} & 0\\ 0 & \mathbb{I}\end{pmatrix}, \gamma^j = \begin{pmatrix}0 & \sigma^j\\ -\sigma^j & 0\end{pmatrix}$

are σ^j Pauli matrices, the Hamiltonian with this notation is

$$H = (-i\hbar c)\left(\alpha_1\partial_{x_1} + \alpha_2\partial_{x_2} + \alpha_3\partial_{x_3}\right) + \gamma^0 mc^2\mathbb{I} \quad (33)$$

and changing the notation of the derivatives for relativistic coordinates, the Dirac Hamiltonian is

$$H_{Dirac} = \left(i\hbar c\gamma^i\partial_i + mc^2\mathbb{I}\right)\gamma^0 \quad (34)$$

Substituting this Hamiltonian in the Schrödinger equation

$$i\hbar c\gamma^0\partial_0\psi = -i\hbar c\gamma^i\partial_i\psi + mc^2\psi \quad (35)$$

Then we arrive at

$$\left(i\hbar c\gamma^\mu\partial_\mu - mc^2\right)\psi = 0 \quad (36)$$

and by choosing the time scale appropriately we can always obtain $\hbar = c = 1$, together with Feynman's slash notation we finally arrive at the Dirac equation

$$\left(i\gamma^\mu\partial_\mu - m\right)\psi = 0 \quad (37)$$

As this equation allows for negative energies, in 1928 Paul Dirac postulated the existence of the antiparticle of the electron, the positron5 The wave functions in Dirac's equation are of a different nature to the wave functions in Schrödinger's equation and are called spinors.

5 The positron was first discovered experimentally in 1932 by Carl David Anderson when photographing the channels of cosmic rays in a cloud chamber (only particles with mass are observed in the cloud chamber).

5. Proposed Model

According to Trzetrzelewski M. (2017), since the emergence of algorithmic trading in the 1980s, changes in prices are generated in a matter of seconds or less, but with the limitation on changes in prices S(t) it cannot be arbitrary in time but there is a speed limit cm , S(t) < cm (known as the speed of light in the market), that is, the speed of information exchange is limited by the velocity of light (30 cm per nanosecond), which for practical purposes makes it very difficult to appreciate these relativistic effects.

On the other hand, a characteristic of liquid markets where the price of an asset goes from S(t) to S (t + dt) within a short period of time introduces a natural concept that we could call friction (or resistance). This situation is similar to electrons moving in semiconductors where an electron can move at an arbitrary speed. Such as the speed light traveling in a medium with density where the speed of light is c/n where n is the refractive index (for example for diamond, water, glass and others).

From what we have reviewed in the previous section, it is known that the Schrödinger equation in imaginary time for a free particle results in a diffusion equation, the correspondence is only formal but has no physical explanation, the mapping $t \to -it$. On the other hand, the mapping suggests that if one wanted to generalize the diffusion equation to the relativistic case one would use the Euclidean version of the Dirac equation as in the following diagram where (v is the speed and c is the speed of light).

Dirac Equation	$\to v \ll c$	Schrödinger Equation
$\downarrow t \to -it$	.	$\downarrow t \to -it$
Euclidian Dirac Equation	.	Euclidian Dirac Equation

An object that satisfies Dirac's equation is a spinor with several components (in our case, (1+1) dimensions, the spinor has 2 real components), but the interpretations of those components are not clear in finance. However, there is a relationship between Dirac's Euclidean equation and stochastic processes.

The generalization in relativity of the original brownian motion theory was recommended in V.A. Fock V.A. (1926) and the original diffusion equation came to be called the damped telegraph wave equation or telegraph equation. however, the first telegraphist equation occurred in 1854 by Lord Kelvin to study the propagation of electrical signals through the transatlantic cable, because the frame of reference of the telegraphist equation and the properties are crucial some important points are mentioned. This topic can review in Trzetrzelewski M.(2018)

The telegrapher's equation is described below, which is very important in the development of the relativistic Brownian motion.

Let be $\xi(x,t)$ an arbitrary function of a set of probability densities $p_+(x,t), p_-(x,t)\ y\ b(x,t)$, then $\xi(x,t)$ is a solution of the so-called telegrapher's partial differential equation in the form

$$\frac{\partial^2 \xi(x,t)}{\partial t^2} + 2\lambda \frac{\partial \xi(x,t)}{\partial t} = v^2 \frac{\partial^2 \xi(x,t)}{\partial x^2} \qquad (38)$$

where $t > 0, x \in \mathbb{R}$ and with initial conditions

$$p_\pm|_{t=0} = \delta(x) \qquad \frac{dp_\pm}{dt}|_{t=0} = \mp v\delta'(x)$$

$$f|_{t=0} = \mathrm{b}|_{t=0} \frac{1}{2}\ \delta(x) \quad \frac{df}{dt}|_{t=0} = -\frac{db}{dt}|_{t=0} = -\frac{v}{2}\delta'(x)$$

Solving the equation for the probability density p(x,t) gives

$$p(x,t) = \frac{e^{-\lambda t}}{2v}[\delta(vt+x)+\delta(vt-x)] + \frac{e^{-\lambda t}}{2v}\left[\lambda I_0\left(\frac{\lambda}{v}\sqrt{v^2t^2-x^2}\right)+\frac{\partial}{\partial t}I_0\left(\frac{\lambda}{v}\sqrt{v^2t^2-x^2}\right)\right]\mathbb{I}_{|x|<vt}$$

(39)

For any $t > 0$, $x \in \mathbb{R}$ where I0(x) of note the modified Bessel function of the form

$$I_0(x) = \sum_{m=0}^{+\infty}\frac{1}{m!\Gamma(m+1)}\left(\frac{x}{2}\right)^{2m} = \sum_{m=0}^{+\infty}\frac{1}{(m!)^2}\left(\frac{x}{2}\right)^{2m} \quad (40)$$

The most outstanding property of the probability distribution function $p(x,t)$for the use of the Goldstein-Kac theory of the telegraph process in the generalization of the Black-Scholes theory which under certain conditions would converge to the probability distribution function of a Wiener process.

The relativistic generalization of the original Black-Scholes theory is connected with two facts. The first of these has to do with the results of high-frequency trading in the market and therefore it is considered important to incorporate the principles of special relativity into the theory. The second point has to do with the volatility parameter in the Black-Scholes model, which is insufficient after what is known as Black Monday in 1987.

According to the concept previously defined as the "speed of market light" cm. It is assumed that if an underlying security has a spot price S(t) at time t, the change in the value of the spot S(t + dt) one time (dt) later (not instantaneous) would follow the following expression

$$v_m \equiv \lim_{\Delta t\to 0}\frac{S(t+\Delta t)-S(t)}{\Delta t} \quad (41)$$

With values less than +∞ and therefore $v_m < +\infty$ but also that it has an upper limit. The upper limit for such a speed is cm and the speed of the effective spot price has an analogy as mentioned above with the movement of electrons moving with an effective speed within a material which is much smaller than the maximum possible velocity that an electron can experience. This value, given by the speed of light in a vacuum c as the signal about the concrete change in the spot price transmitted at the speed of light in a vacuum and can also be interpreted as the desire of investors to buy or sell an underlying asset, is as follows

$$\left|\frac{dx(t)}{dt}\right| = \frac{\left|\frac{dS(t)}{dt}\right|}{S(t)} < \frac{c_m}{S(t)} < 1 \quad (42)$$

The objective is to find a probability distribution $p_r(x,t)$ that will be used as an integration kernel (that is, a financial propagator) in a Feymann-Kac risk-neutral valuation formula, that is

$$C(S(t),t) = e^{-r(T-t)}\int_{-\infty}^{+\infty} f(x(T),T|x(t),t)g(x(T))dx(T) \quad (43\ a)$$

or in another variable of the same spot price $S(t) = e^x$ as

$$C(S(t),t) = e^{-r(T-t)}\int_{0}^{+\infty} f(S(T),T|S(t),t)g(S(T))\frac{dS(T)}{S(T)} \quad (43\ b)$$

which we can rewrite

$$C(S(t),t) = e^{-r(T-t)}\int_{0}^{+\infty} f\big(x\big(S(t)\big),T-t\big)g(S(T))\frac{dS(T)}{S(T)} \quad (43\ c)$$

where

$$x\big(S(T)\big) = ln\left(\frac{S(t)}{S(T)} - +\left(r-\frac{1}{2}\sigma^2\right)(T-t)\right) \quad (44)$$

the relativistic generalization of Schrödinger's equation, as mentioned above, is Dirac's equation in 1 dimension for position and 1 dimension for time. In the following, it will be shown that

that equation is equivalent to Kolmogorov's equations from the previous section and therefore equivalent to the telegrapher's equation.

For another hand, following Blaho R. (2021) Let us remember that Dirac's equation (eq. 37) in (1+1) dimension has the following expression

$$\left(i\hbar\gamma^{\mu}\partial_{\mu} - m_0 c\right)\psi(x,t) = 0 \tag{45}$$

with γ^{μ} the gamma matrices, however, in this case the Greek indices represent a two-dimensional analogy to four-dimensional Minkowski space i.e. $\mu \in \{0,1\}$ developing the notation we obtain and multiplying by the speed of light (c) in a vacuum.

$$(ic\hbar\gamma^{0}\partial_{0} + ic\hbar\gamma^{1}\partial_{1} - m_0 c^2)\psi(x,t) = 0 \tag{46}$$

finally we are developing the gradient notation and with the intention of obtaining the Dirac equation (1+1), the equation is multiplied by the matrix $\gamma^{0^{-1}}$ by and accommodating terms

$$i\hbar\frac{\partial\psi(x,t)}{\partial t} = m_0 c^2\gamma^{0^{-1}}\psi(x,t) - ic\hbar(\gamma^{0^{-1}}\gamma^{1})\frac{\partial\psi(x,t)}{\partial x} \tag{47}$$

that have the same form as the Dirac equation (1+1) that can be seen in the previous sections. This derivation is crucial, which is not explicitly done from both sources, the last but not least important thing is to establish the following correspondence

$$\gamma^{0^{-1}} = \sigma_1\,, \gamma^{0^{-1}}\gamma^{1} = \sigma_3$$

to obtain the equation as it can be read with the Pauli matrices in the form

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and equation of (1+1) Dirac has the form of the

$$i\hbar\frac{\partial\psi(x,t)}{\partial t} = \left(-ic\hbar\sigma_3\frac{\partial}{\partial x} + \sigma_1 m_0 c^2\right)\psi(x,t)(x,t) \tag{48}$$

with the Dirac spinor

$$\psi(x,t) = \begin{pmatrix} \psi_+(x,t) \\ \psi_-(x,t) \end{pmatrix} \tag{49}$$

it must be verified that the relations satisfied with Clifford algebra, if one exploits the relations, the multiplication of simple matrices gives $\{\gamma^{\mu},\gamma^{\nu}\} = 2g^{\mu\nu}\mathbb{1}_{2x2}$

$$\gamma^{0} = \sigma_1 \tag{50 a}$$

$$\gamma^{1} = \sigma_1\sigma_3 \tag{50 b}$$

using the Pauli matrices given the results, now then the Dirac equation in (1+1) dimensions can be obtained, one can change the variables to be able to get the telegraph equation, using

$$u(x,t) = e^{\frac{im_0c^2t}{\hbar}}\psi(x,t) \quad \text{o bien} \quad \psi(x,t) = e^{\frac{-im_0c^2t}{\hbar}}u(x,t) \tag{51}$$

We insert this expression of the coupled (1+1) Dirac equation for the system for each component of Dirac spinor in the form

$$i\hbar\frac{\partial\psi_+(x,t)}{\partial t} = -ic\hbar\frac{\partial\psi_+(x,t)}{\partial x} + m_0c^2\psi_-(x,t) \tag{52 a}$$

$$i\hbar\frac{\partial\psi_-(x,t)}{\partial t} = +ic\hbar\frac{\partial\psi_-(x,t)}{\partial x} + m_0c^2\psi_+(x,t) \tag{52 b}$$

result

$$\frac{\partial u_+(x,t)}{\partial t} = -c\frac{\partial u_+(x,t)}{\partial x} + \frac{im_0c^2}{\hbar}(u_+(x,t) - u_-(x,t)) \tag{53 a}$$

$$\frac{\partial u_-(x,t)}{\partial t} = +c\frac{\partial u_-(x,t)}{\partial x} + \frac{im_0c^2}{\hbar}(u_-(x,t) - u_+(x,t)) \tag{53 b}$$

which are the Kolmogorov equations of Goldstein-Kac's theory of the telegraph process with imaginary intensity coefficient given by $\lambda = -\frac{im_0 c^2}{\hbar}$, the velocity of the particle denoted by c, the point is that the intensity of the stochastic process should be a real number instead of an imaginary number. This can be resolved when additional transformations are introduced, that is, the transformation of time and Wick's rotation $\bar{t} = it$ together with the Euclidean introduction of the speed of light $\bar{c} = ic$ and transiting in the Euclidean world. If one applies this notation, one obtains

$$\frac{\partial u_+(x,\tilde{t})}{\partial \tilde{t}} = -\tilde{c}\frac{\partial u_+(x,\tilde{t})}{\partial x} + \frac{m_0\tilde{c}^2}{\hbar}(u_+(x,\tilde{t}) - u_-(x,\tilde{t})) \quad (54\ a)$$

$$\frac{\partial u_-(x,\tilde{t})}{\partial \tilde{t}} = +\tilde{c}\frac{\partial u_-(x,\tilde{t})}{\partial x} + \frac{m_0\tilde{c}^2}{\hbar}(u_-(x,\tilde{t}) - u_+(x,\tilde{t})) \quad (54\ b)$$

we can see the equivalence between Dirac's equation (1+1) and Kolmogorov's equation with the telegrapher's equations and there is equivalence between Schrödinger's equations and the classical diffusion equation. To complete we write of telegraphers for each component separately

$$-\frac{\partial^2 u_+(x,t)}{\partial t^2} - c^2\frac{\partial^2 u_+(x,t)}{\partial x^2} = +2\frac{im_0c^2}{\hbar}\frac{\partial u_+(x,t)}{\partial t} \quad (55\ a)$$

$$-\frac{\partial^2 u_-(x,t)}{\partial t^2} - c^2\frac{\partial^2 u_-(x,t)}{\partial x^2} = +2\frac{im_0c^2}{\hbar}\frac{\partial u_-(x,t)}{\partial t} \quad (55\ b)$$

in the limit $c \to +\infty$ one recovers the classical diffusion equation equivalent to Schrödinger's equation for example with the component $u_+(x,t)$one obtains

$$-\frac{\partial^2 u_+(x,t)}{\partial x^2} = +2\frac{im_0}{\hbar}\frac{\partial u_+(x,t)}{\partial t} \quad (56)$$

finally dividing the last equation by 2m0 together with the multiplication of $\hbar^2$ we have

$$-\frac{\hbar^2}{2m_0}\frac{\partial^2 u_+(x,t)}{\partial x^2} = +i\hbar\frac{\partial u_+(x,t)}{\partial t} \quad (57)$$

which is a non-relativistic diffusion equation with arranged coefficients (again with an imaginary diffusion coefficient one would simply do Wick rotation) and again incorporating the wave equation of the equivalent probability density one obtains

$$-\frac{\hbar^2}{2m_0}\frac{\partial^2 \psi_+(x,t)}{\partial x^2} = +i\hbar\frac{\partial \psi_+(x,t)}{\partial t} \quad (58)$$

which is the non-relativistic Schrödinger equation of quantum mechanics, if the equivalence between the (1+1) dimensional Dirac equation for particles with spin 0 and the telegraph equation is used, through the already known Kolmogorov equation, the original normal Gaussian distribution of a Wiener process is obtained as a solution of the classical diffusion equation which handles the evolution of the underlying asset price of the Black Scholes equation model having to be substitutes for

$$p_r(x,t) = \frac{e^{-\lambda t}}{2c_m}[\delta(c_m t + x) + \delta(c_m t - x)] + \frac{e^{-\lambda t}}{2c_m}\left[\lambda I_0\left(\frac{\lambda}{c_m}\sqrt{c_m^2 t^2 - x^2}\right) + \frac{\partial}{\partial t} I_0\left(\frac{\lambda}{c_m}\sqrt{c_m^2 t^2 - x^2}\right)\right]\mathbb{I}_{|x|<c_m t} \quad (59)$$

this equation is already relativistic and is in agreement with the telegraph equation it does not allow the particle in higher positions $x > c_m t$ where c_m denotes the speed of light in the market. Using the transformation of the variable in the limit $\lambda = \frac{c_m^2}{\sigma^2}$ and with an asymptotic expansion one arrives at the following representation $p(x, T-t)$

$$p_r(x, T-t) \approx \frac{e^{-\frac{x^2}{2\sigma^2(T-t)}}}{\sqrt{2\pi\sigma^2(T-t)}}\left(1 + \frac{1}{c_m^2} f_r(x, T-t)\right) \quad (60)$$

with

$$f_r(x,T-t)=-\frac{\sigma^2}{8(T-t)}+\frac{x^2}{2(T-t)}-\frac{x^4}{8\sigma^2(T-t)^3} \quad (61)$$

the easiest way to see that the subsequent expansion is valid following the following fact

$$I_0(a)\approx\frac{e^a}{\sqrt{2\pi a}}\left(1+\frac{1}{8a}\right) \quad (62)$$

Keeping for $a\gg 1$the argument $\frac{\lambda}{c_m}\sqrt{c_m^2t^2-x^2}=\sqrt{1-\frac{x^2}{tc_m^2}}$ in the limit $c_m\to+\infty$ $\lambda\to+\infty$ in certainly greater using the Taylor expansion we obtain the last expression which is to consider the function

$$\frac{\lambda}{c_m}\sqrt{c_m^2t^2-x^2}\approx\lambda t-\frac{\lambda x^2}{2c_m^2t}-\frac{\lambda x^4}{8c_m^4t^3} \quad (63)$$

the way in which one could use the Taylor expansion to obtain the function $f(a)=\sqrt{1-a}$, the second order Taylor expansion for this function in the variable around the point a = 0 one has

$$f(a)\approx 1-\frac{a}{2}-\frac{a^2}{8} \quad (64)$$

substituting gives $a=\frac{x^2}{tc_m^2}$ the desired result, the reason for why one takes the expansion coefficient putting the second order in the effect, which higher order terms does not include the speed of light in the market c_m only in the denominator and this is not important. Then the Black Scholes formula generalization with the subsequent expansion of the probability distribution of the telegraph operator has the form

$$C(S(t),t)=\frac{Ke^{-r(T-t)}}{\sqrt{2\pi\sigma^2(T-t)}}\int_0^{y_{max}}\left((e^y-1)e^{-\frac{(x-y)^2}{2\sigma^2(T-t)}}\right)\left(1-\frac{1}{c_m^2}f(y,T-t)\right)dy \quad (65)$$

$$x(S(t))=\ln\frac{S(t)}{K}+\left(r-\frac{1}{2}\sigma^2\right)(T-t) \quad (66)$$

where we change the integration variables for convenience, the upper integration limit and ymax max is implicitly given by which $f(y_{max},\tau)=c_m^2$ has four solutions, however only one of them is real and positive

$$y_{max}=\sqrt{2\sigma^2(T-t)+\sigma(T-t)\sqrt{3\sigma^2+8c_m^2(T-t)}} \quad (67)$$

at the limit $c_m\to\infty$ we have $y_{max}\to\infty$ and the integral converges to the Black Scholes expression. In this case c_m the integral becomes a little more complicated because the exponential dampens the integrand. We can approximate the integral by assuming that $y_{max}=\infty$, , then the negligible error compared to the corrections will be introduced $1/c_m^2$. The final result is relatively simple in terms of standard parameters d1 and d2 known

$$C(S(t),T-t)=SN(d_1)-Ke^{-r(T-t)}SN(d_2)+\frac{1}{c_m^2}\upsilon \quad (68)$$

$$d_1=\frac{\sigma^2(T-t)+x}{\sigma\sqrt{T-t}},\ d_2=\frac{x}{\sigma\sqrt{T-t}} \quad (69)$$

$$\upsilon=-\frac{\sigma^2}{8(T-t)}\left[SM(d_1)-Ke^{-r(T-t)}M(d_2)\right]-$$

$$\frac{S\sigma^2}{8\sqrt{2\pi(T-t)}}e^{-\frac{d_1^2}{2}}\left(1+\frac{3}{2}d_1^2+\frac{3}{2}d_2^2-\frac{1}{2}\sigma^2(T-t)\right)$$

where

$$M(z):=N(z)z^2(z^2+2),\quad N(z)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z}e^{-\frac{t^2}{2}}dt \quad (70)$$

6. Results

In the previous section, we reviewed the generalization of the Black-Scholes equation to the relativistic version and where it requires the use of concepts and techniques developed within quantum and relativistic physics, the physical and mathematical ideas and techniques within their own disciplines represent a challenge in their understanding and application, even more so within a completely different area.

From equations (68), (69) and (70) it can be seen that the solution for determining the premium of a call option is very similar to the classical model but with a correction term associated with the speed of information transfer or speed of light in the market.

If this transmission speed is very close to the speed of light and the distances involved are on a terrestrial scale, the effects will be practically almost imperceptible, that is to say, if and the correction term disappears.

However, the term cm (the speed of information) does not only involve the speed of light itself, but also the medium in which it is transmitted, considering fiber optics, microwaves, lasers and communication via the internet and within computers themselves. This speed of light in the market can give small advantages in financial trading and generate arbitrage opportunities.

Below we made an exercise in which we assume that the prices of financial options in the market are known and we assume a percentage lower than their value (due to the delay in information) and we look for the implicit value of the speed of light in the market that it would have to have to generate that price. Below are the results for the call and put in an exercise that is done for certain values, without losing generality.

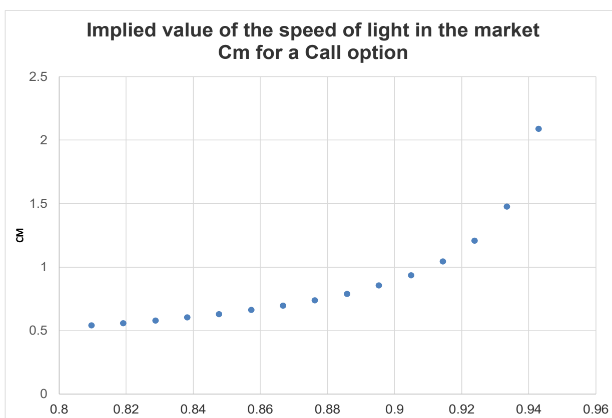

Figure 1. Implied value of the speed of light cm for a Call option (S=18, K=20, T=1, r = 0.04, σ=0.20)
Source: Own elaboration

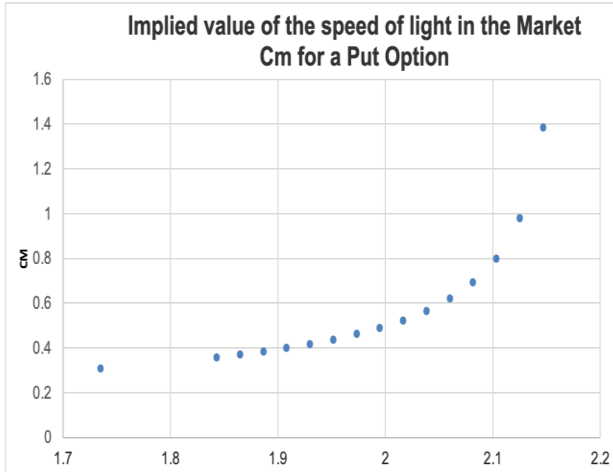

Figure 1 Implied value of the speed of light cm for a Put option (S=18, K=20, T=1, r=0.04, σ=0.20)
Source: Own elaboration

4. Conclusions

This section presents the main conclusions of this work:

Although the ideas of hedging against market uncertainty are old, it was not until the arrival of the Black-Scholes equation that it was considered a consistent and practical methodology for the valuation of call and put type hedging options. The results have been extended in various directions including fitting models with Poisson jumps, Levy processes, volatility adjustments models, always with the aim of improving their valuation estimates of these derivatives, the work presents a relativistic Black Scholes equation that has been well known.

To extend the Black Scholes concepts to include the ideas of special relativity, it is necessary to incorporate ideas from physics such as quantum mechanics and relativistic mechanics so that together and based on Dirac's equation, a path can be found to construct an adequate probability density.

For the corrections in the estimations to be really significant in Black Scholes premium, there would practically have to be interplanetary trading where the information lag was significant (for example, sunlight takes 8 minutes to reach the earth). However, the speed of light in the market can have a representation or equivalent in the delay of information coming from communication technology, such as optical fibers, laser communication, satellites or simply the internet and even the time in human reactions. So that tiny advantage in time could be converted into a window of opportunity with certain high-frequency trading algorithms.

With the concept of the cm parameter, once determined, it can be implicitly applied to delays arising from information, not only due to physical distances but also due to technological factors and human reactions and platforms, and it can be a characteristic of each market and region.

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